Lecture 5

This document provides practice problems that are similar to those that will be asked during the final exam. Please note that the document reflects the style and not the number of the questions that will be on the exam.

Problem 1

Consider the scalar continuous-time dynamical system described by $\dot{x} = \cos(x)$. What can we say about its equilibria?

- **(a)** The points $\{x_k = \pi/2 + \pi k, k \in \mathbb{Z}\}$ are unstable equilibria. **[False]**
- **(b)** The points $\{x_k = \pi/2 + \pi k, k \in \mathbb{Z}\}$ are stable equilibria. **[False]**
- **(c)** The points $\{x_k = \pi/2 + 2\pi k, k \in \mathbb{Z}\}$ are unstable equilibria. The points $\{x_k = (3/2)\pi + 2\pi k, k \in \mathbb{Z}\}$ are stable equilibria. **[False]**
- **(d)** The points $\{x_k = \pi/2 + 2\pi k, k \in \mathbb{Z}\}$ are stable equilibria. The points $\{x_k = (3/2)\pi + 2\pi k, k \in \mathbb{Z}\}$ are unstable equilibria. **[True]**
- **(e)** The points $\{x_k = \pi/2 + \pi k, k \in \mathbb{Z}\}$ are not equilibrium points. **[False]**

Problem 2

Consider the scalar continuous-time dynamical system described by $\dot{x} = \cos(x)$. What can we say about a trajectory that starts at the origin at time $t = 0$?

- **(a)** The trajectory is not unique. **[False]**
- **(b)** The trajectory oscillates between $-\pi/2$ and $\pi/2$. **[False]**
- **(c)** The trajectory approaches $\pi/2$ for $t \longrightarrow \infty$. **[True]**

  Explanation: Solving the differential equation yields $x(t) = 2\tan^{-1}(\tanh(t/2))$. We note that $\tanh(t/2) \longrightarrow 1$ for $t \longrightarrow \infty$, and $\tan^{-1}(1) = \pi/4$. However, for this problem it suffices to argue quantitatively, since we are dealing with a first-order differential equation in one dimension: Between 0 and $\pi/2$, $\cos(x)$ is positive, meaning that $x(t)$ will increase. Moreover, $\pi/2$ is a stable equilibrium (see problem 1), thus a trajectory starting from $x(0) = 0$ will increase and converge to $\pi/2$.
- **(d)** The trajectory approaches $-\pi/2$ for $t \longrightarrow \infty$. **[False]**

Problem 3 (answer corrected on December 8, 2021)

Are there limit points of continuous-time dynamical systems that are not equilibria?

- **(a)** No. **[False]**
- **(b)** Yes. **[True]**

  Explanation: Consider, for example, periodic orbits.

Problem 4

Consider the scalar continuous-time dynamical system $\dot{x} = -0.5x - \sin(x)$. Which of the following statements are correct?

(a) $V(x) = 0.25x^2 - \cos(x) + 1$ is a Lyapunov function for this system. **[True]**

(b) $V(x) = -0.25x^2 + \cos(x) - 1$ is a corresponding Lyapunov function. **[False]**

Explanation: $-\mathrm{d}(V(x(t)))/\mathrm{d}t$ is not positive definite.

(c) The equilibrium at the origin is unstable. **[False]**

Problem 5

We analyze gradient-flow dynamics on the objective function $-\cos(x)$, which leads to the dynamical system $\dot{x} = -\sin(x)$. Does the following claim on the convergence rate hold?

$$|x(t)| \leq 10 \cdot |x(0)| \cdot \exp(-t) \quad \forall t \geq 0 \text{ and } \forall x(0) \in (-\pi, \pi).$$

(a) Yes. **[False]**

(b) No. **[True]**

Explanation: A trajectory starting close to $-\pi$ or π takes an arbitrarily long time to converge to the origin.

Problem 6

We recall that the space L_{2e} is defined as

$$L_{2e} := \left\{ g : [0, \infty) \to \mathbb{R} \;\middle|\; \int_0^T g(t)^2 \, \mathrm{d}t < \infty, \; \forall T \geq 0 \right\}.$$

Which of the following statements are true?

(a) The function $f(t) = e^t$ is an element of L_{2e}. **[True]**

Explanation: The value of the the integral is $e^T \sinh T$, which is finite for finite positive T.

(b) The function $f(t) = t^3$ is an element of L_{2e}. **[True]**

Explanation: The value of the the integral is $T^7/7$, which is finite for finite positive T.

(c) The function $f(t) = 1/(t-1)$ is an element of L_{2e}. **[False]**

Explanation: The integral does not converge for $T \geq 1$.

(d) The function $f(t) = 1/|t-1|^{1/4}$ is an element of L_{2e}. **[True]**

Explanation: The value of the integral is $-2\sqrt{1 - \min(1, T)} + 2\sqrt{T-1} - 2\sqrt{\min(1, T) - 1} + 2$, which is finite for finite positive T.

In practice, it is useful to distinguish the cases $T \leq 1$ and $T > 1$ and use the linearity of the integral: $\int_0^T h(t)\,\mathrm{d}t = \int_0^1 h(t)\,\mathrm{d}t + \int_1^T h(t)\,\mathrm{d}t$ for $T > 1$ to find that the integral converges in both cases, which is sufficient to answer the question.

Note: Assume that all functions f here are restricted to the domain $[0, \infty)$.