# Lecture 2

This document provides practice problems that are similar to those that will be asked during the final exam. Please note that the document reflects the style and not the number of the questions that will be on the exam.

## Problem 1

Consider the following zero-sum game, where two players $P_1$ and $P_2$ can choose between actions $A$ and $B$, and receive the payoff according to the following table:

| | | $P_2$: A | $P_2$: B |
|---|---|---|---|
| $P_1$ | A | 3 | 1 |
| | B | 4 | 5 |

For example, if $P_1$ selects action $A$ and $P_2$ selects action $B$, then $P_1$ receives reward 1, while $P_2$ receives reward $-1$.

Which of the following statements are correct?

**(a)** The action profile where $P_1$ chooses $A$ and $P_2$ chooses $B$ corresponds to a Nash equilibrium. **[False]**

**(b)** The action profile where $P_1$ chooses $B$ and $P_2$ chooses $B$ corresponds to a Nash equilibrium. **[False]**

**(c)** The action profile where $P_1$ chooses $B$ and $P_2$ chooses $A$ corresponds to a Nash equilibrium. **[True]**

**Explanation:** Given that $P_2$ plays $A$, $P_1$ has no incentive to deviate from $B$. Similarly, given that $P_1$ plays $B$, $P_2$ has no incentive to deviate from $A$.

**(d)** The game has only a Nash equilibrium if the two players are allowed to play mixed strategies. **[False]**

## Problem 2 (answers updated on December 20, 2021)

Consider the following zero-sum game where players $P_1$ and $P_2$ can choose between actions $A$ and $B$ and receive a payoff according to the following table:

| | | $P_2$: A | $P_2$: B |
|---|---|---|---|
| $P_1$ | A | 0.5 | 1 |
| | B | 3 | $x$ |

where $x \in \mathbb{R}$. Which of the following statements are correct?

**(a)** There exists a Nash equilibrium with mixed strategies for any $x < 1$. **[True]**

**(b)** For all $x \in \mathbb{R}$ there exists a unique Nash equilibrium. **[False]**

**Explanation:** See discussion on Moodle: https://moodle-app2.let.ethz.ch/mod/forum/discuss.php?d=94366.

**(c)** If $P_2$ plays according to the (Nash) equilibrium strategy, their strategy will be pure for $x \geq 1$. **[True]**

*Hint:* Sketch the expected reward for both players, as we did in the lecture.

## Problem 3 (question and answers updated on November 17, 2021)

Let $A \in \mathbb{R}^{2\times 2}$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{11} + c & a_{12} + c \end{pmatrix},$$

describe the rewards of a two-player zero sum game. For example, if Player 1 plays action 1 and Player 2 plays action 2, Player 1 receives reward $a_{12}$, whereas Player 2 receives reward $-a_{12}$. Both players play according to Nash equilibrium strategies.

Which of the following conditions are true for arbitrary $a_{11}, a_{12}, a_{11} \neq a_{12}$, and $c \neq 0$?

**(a)** Player 1 has a pure strategy. **[True]**

**Explanation:** If $c > 0$, Player 1 always plays action 2; if $c < 0$, Player 1 always plays action 1.

**(b)** Player 1 has a strictly mixed strategy. **[False]**

**(c)** Player 2 has a strictly mixed strategy. **[False]**

**Explanation:** Similar reasoning as for Player 1.

**(d)** None of the above. **[False]**

## Problem 4

Let $A \in \mathbb{R}^{2\times 2}$ be given as

$$A = \begin{pmatrix} 0.5 & 1 \\ 2 & 0.5 \end{pmatrix},$$

and let

$$x^* := \arg\max_{x\in\Delta} \left( \min_{y\in\Delta} \left(x^\top A y\right)\right), \qquad y^* := \arg\min_{y\in\Delta} \left( \max_{x\in\Delta} \left(x^\top A y\right)\right),$$

where $\Delta$ denotes the two-dimensional unit simplex, that is, $\Delta := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1\}$. Which of the following results is correct?

**(a)** $x^* = (3/4, 1/4), \quad y^* = (1/4, 3/4)$. **[True]**

**Explanation:** There are multiple ways how to arrive at this solution:

*Version 1:* Let $x = \begin{pmatrix} p & 1-p \end{pmatrix}^\top$ and $y = \begin{pmatrix} q & 1-q \end{pmatrix}^\top$. In this case, $x^T A y = 0.5 + 0.5p + 1.5q - 2pq =: f(p, q)$. To find the critical points of this function, we compute: $\partial/\partial p\, f = 0.5 - 2q \stackrel{!}{=} 0 \Rightarrow q = 1/4$, and $\partial f/\partial q = 1.5 - 2p \stackrel{!}{=} 0 \Rightarrow p = 3/4$.

*Version 2:* We recognize that solving the saddle point problem is equivalent to finding the (mixed-strategy) Nash equilibrium for a zero-sum game with payoff matrix $A$. Let the two strategies be parametrized as $x = \begin{pmatrix} p & 1-p \end{pmatrix}^\top$ and $y = \begin{pmatrix} q & 1-q \end{pmatrix}^\top$. To find the optimal strategy $x^*$ for player 1, we look at her expected payoff. If player 2 plays action 1, the expected payoff for player 1 is $0.5p + 2(1-p)$; if player 2 plays action 2, the expected payoff for player 1 is $p + 0.5(1-p)$. Since player 1 optimizes the worst case, she chooses $p$ such that $0.5p + 2(1-p) \stackrel{!}{=} p + 0.5(1-p)$, which yields $p = 3/4$. An analogous argument for player 2 yields $q = 1/4$.

**(b)** $x^* = (1/4, 3/4), \quad y^* = (3/4, 1/4)$. **[False]**

**(c)** $x^* = (2/3, 1/3), \quad y^* = (1/4, 3/4)$. **[False]**

**(d)** $x^* = (1, 0), \quad y^* = (0, 1)$. **[False]**

**(e)** None of the above. **[False]**

## Problem 5

Is the following statement correct: "Any two-player game with a finite number of actions admits a Nash equilibrium with mixed strategies"?

**(a)** Yes. **[True]**

**Explanation:** This result is called Nash's Existence Theorem and was proven in the lecture.

**(b)** No. **[False]**

## Problem 6

Consider a zero-sum game with two players and a finite number of actions which has a mixed Nash equilibrium. Is this equilibrium necessarily unique?

**(a)** Yes. **[False]**

**(b)** No. **[True]**

**Explanation:** Consider the counter-example of a game with constant payoff 1 for player 1 and -1 for player 2 (for every combination of actions).