Brief overview of dynamical systems and control

- Literature
  - S.H. Strogatz, "Nonlinear dynamics and chaos", Perseus Books, 1994
  - S. Wiggins, "Introduction to applied nonlinear dynamical systems and chaos", 2nd edition, Springer, 2000
  - S. Sastry, "Nonlinear Systems: Analysis, Stability and Control", Springer, 1999
  - H.K. Khalil, "Nonlinear Systems", 3rd edition, Prentice Hall, 2002

1. Definition

- Continuous-time dynamical system is given by

$$\dot{x}(t) = f(x(t)), \quad t \in \mathbb{R}. \tag{1}$$

- Discrete-time dynamical system is given by

$$x_{k+1} = F(x_k), \quad k \in \mathbb{Z}.$$

- f is Lipschitz continuous, F is continuous.
- $x(t)$, x_k is called the "state" of the system → knowledge of $x(t_0)$, x_{k_0} allows to compute $x(t)$ for $t \geq t_0$, x_k for $k \geq k_0$

2. Motivation

i) many learning problems involve dynamical systems

↳ Cubli: → adapting to wear and tear in brakes / adapting to changing mass and sensor biases

↳ Pamy: → friction, muscle-based actuation is difficult to model from 1st principles

ii) learning algorithms can be viewed as dynamical systems

→ online learning → minimize

$$E_{XY}\left[\, l(Y, f(X,\alpha))\right] \qquad \text{(see lect. 1)}$$

with stochastic gradient descent:

$$\underbrace{\alpha_{k+1}}_{} = \alpha_k - T_k \; \nabla_\alpha l(Y_k, f(X_k, \alpha_k))$$

state of dyn. system — α_k

step-size — T_k

samples — Y_k, X_k (may be dependent on α_k)

→ if l is convex in α and (X_k, Y_k) are indep. samples we can prove $1/\sqrt{k}$ convergence-rates (c.f. lecture 1)

→ if l is strongly convex in α and (X_k, Y_k) are indep. samples we can prove $1/k$ convergence-rates.

2. Basic Definitions

- Plot trajectories in the phase space

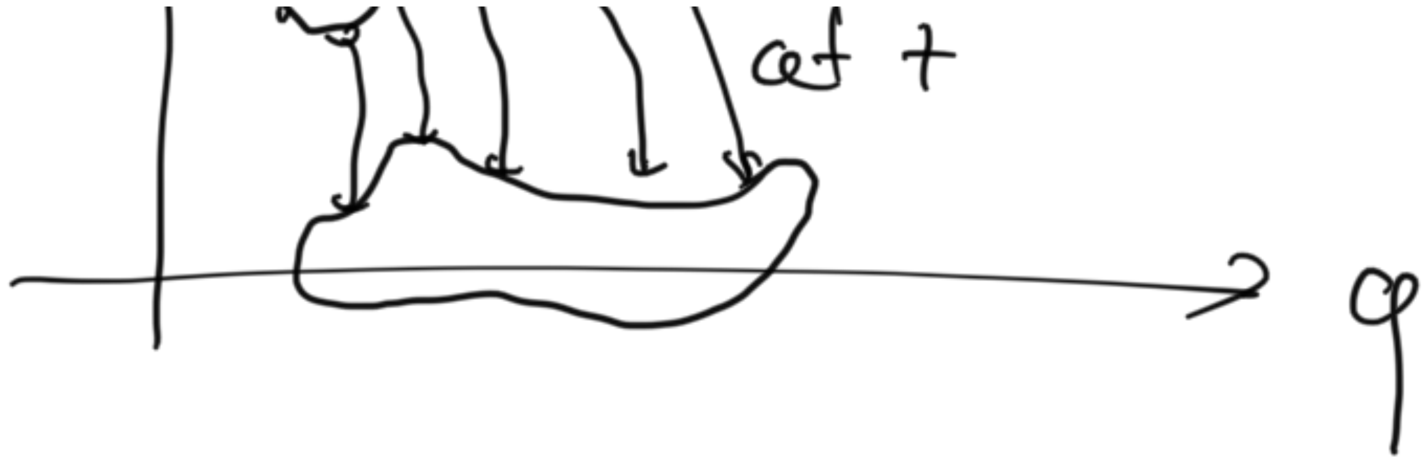
$$\dot q = p$$
$$\dot p = -2dp - \nabla f(q)$$

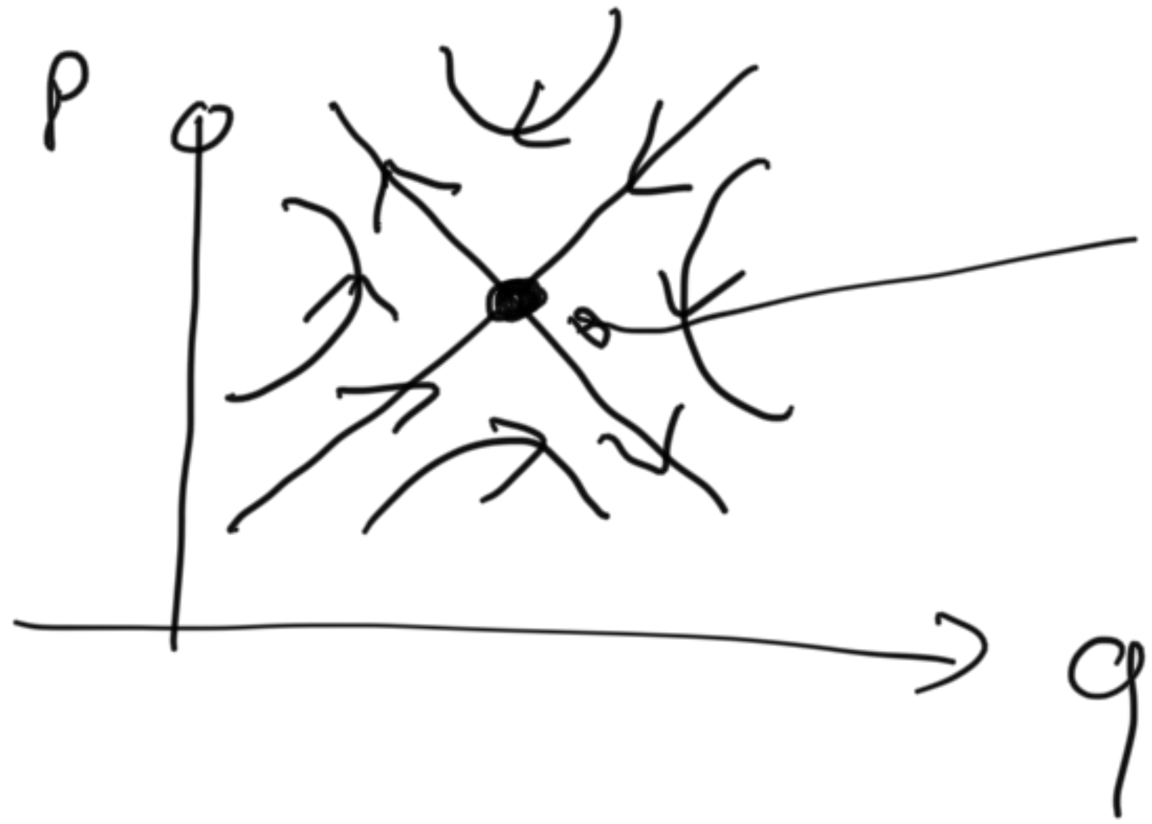
- For $x(t_0) = x_0$, (1) describes a trajectory for $t \in \mathbb{R}$, $x(t) = \varphi(t, t_0, x_0)$

φ → flow of the dynamical system

t_0: initial time

x_0: initial cond



- 

equilibrium

→ trajectories do not cross!

- limit point: $p \in \mathbb{R}^n$ is a limit point if

there exists a sequence

$t_1 < t_2 < t_3 < \dots$ (or $t_1 > t_2 > \dots$)

such that $\lim_{k \to \infty} t_k = \infty$

(or $\lim_{k \to \infty} t_k = -\infty$) such that

$k \to \infty$

$$\lim_{k\to\infty} \varphi(t_k, t_0, x_0) = p.$$

- $\underline{x_n = x_{n-1}\, r\,(1 - x_{n-1})}$, $\quad r \in \mathbb{R}$

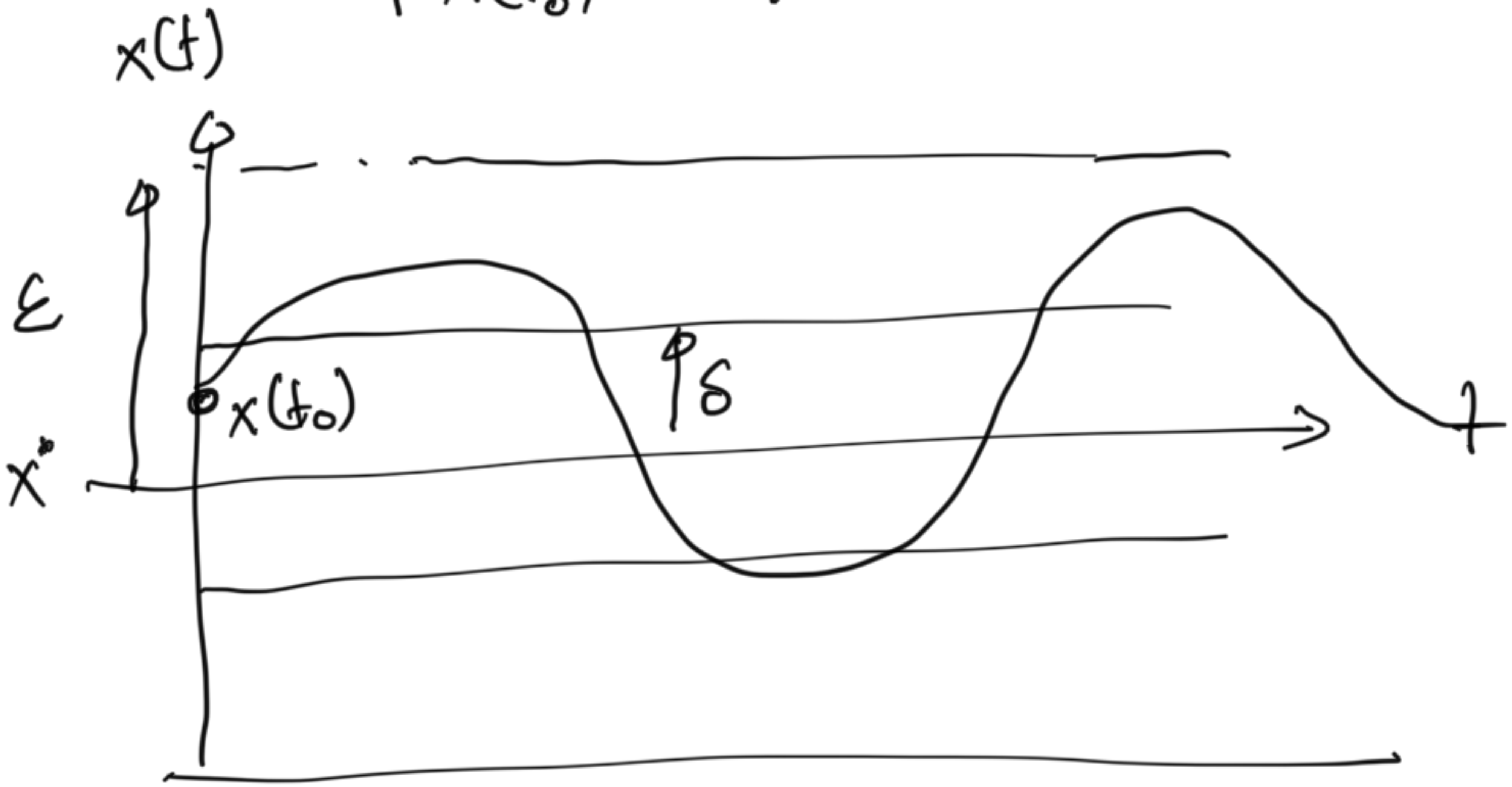
- limit set is the set of all limit points
  - ↳ equilibria
  - ↳ limit-cycles
  - ↳ strange attractor (chaos)

3. Lyapunov Stability of Equilibria (c.t. dynamics)

Def.: An eq. x^* $(f(x^*)=0)$ of (1) is Lyap.

stable if for each $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|x(t_0) - x^*| < \delta \implies |x(t) - x^*| < \varepsilon \quad \forall t \geq t_0 .$$



→ The def. is equiv. to saying that $\varphi(t, t_0, x_0)$ is continuous at x^*, uniformly in time.

... is locally attractive if there

• Def. An eq. is [illegible]
exist $\delta > 0$ such that

$$|x(t_0) - x^*| < \delta \Rightarrow \lim_{t\to\infty} \phi(t, t_0, x_0) = x^*$$

• Def. An eq. is asympt. stable if it is stable and attractive.

• How to check stability?

• Lyapunov's direct method

• Def. $v : \mathbb{R}^n \to \mathbb{R}$ is p.d. if it is continuous, if $v(0) = 0$, if $v(x) > 0$ $\forall x \neq 0$, $|x| < h$.

example:

• <u>Thm.</u> : • WLOG we assume that $x = 0$.

• if there exists a p.d. function v and $h > 0$ such that

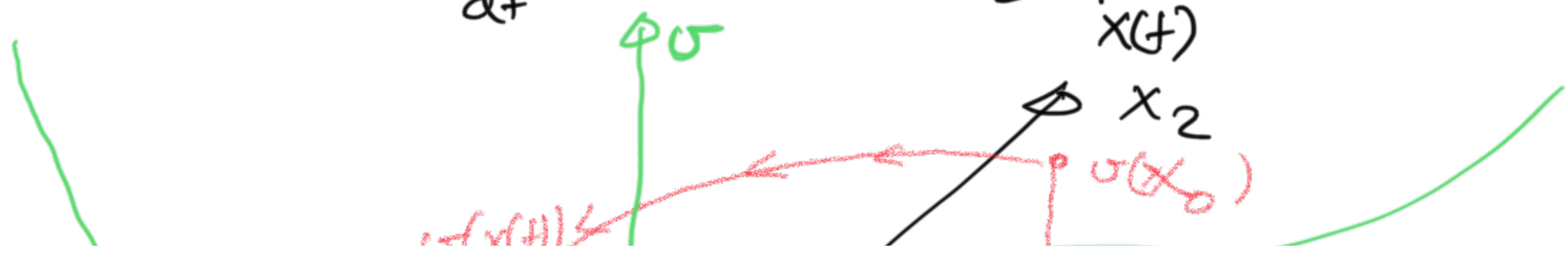
$$\bullet \quad \frac{\partial v}{\partial x} f(x) \leq 0 \qquad \forall x : |x| \leq h$$

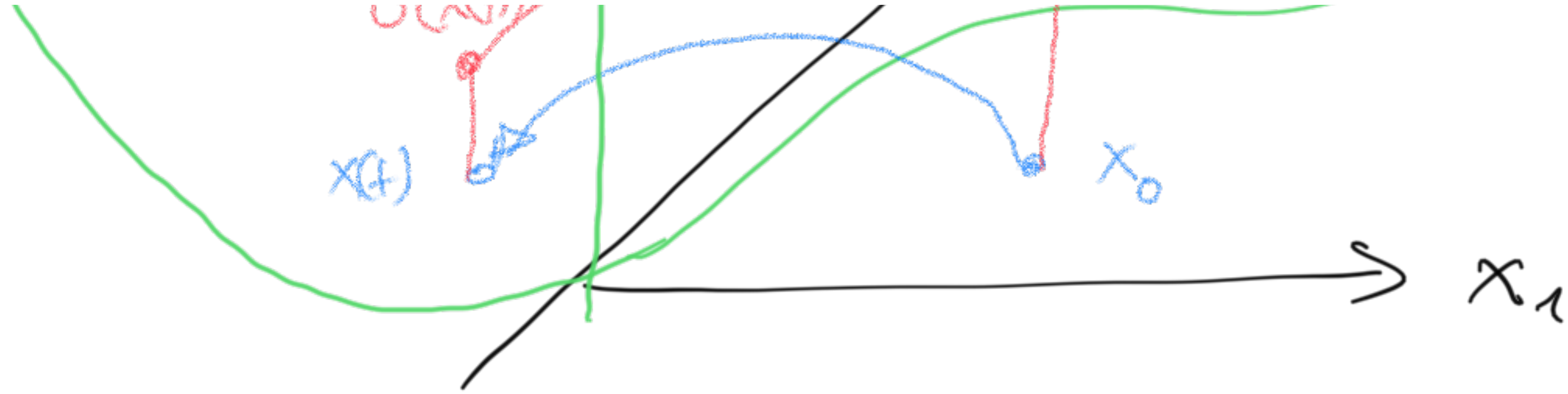
$\rightarrow$ $x^* = 0$ is a stable eq.

• if in addition $-\frac{\partial v}{\partial x} f(x)$ is p.d. (for $|x| \leq h$) then 0 is an asymp. stable eq.

• <u>Idea</u> : $\rightarrow$ analyze how $v(x(t))$ evolves.

$$\frac{d}{dt} v(x(t)) = \left.\frac{\partial v}{\partial x}\right|_{x(t)} f(x(t)) \leq 0$$





- **Thm.** if $x^* = 0$ is an asympt. stable eq., $\exists$ a p.d. function σ, that satisfies G).

- You can choose $z(t) = x(t)\, e^{\rho t}$, $\rho > 0$ fixed

  $\rightarrow \quad \dot{z} = f(z e^{-\rho t})\, e^{\rho t} + \rho\, z(t)$

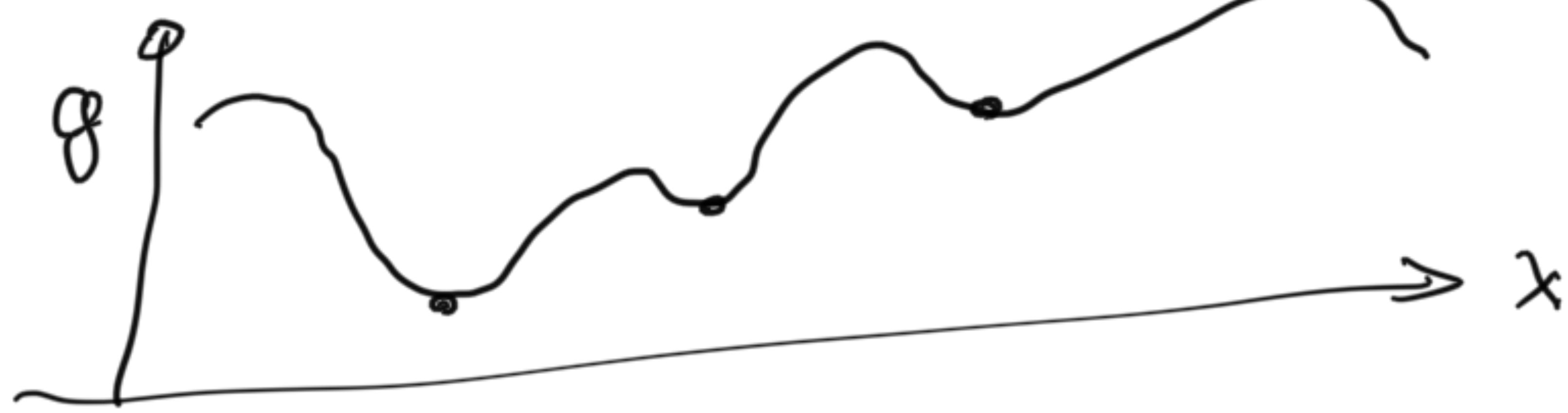
  $\rightarrow \quad z \rightarrow 0$ is stable for the z-dynamics

  $\Rightarrow$ x converges with rate ρ.

- **Example.** Gradient flow: $\quad \dot{x} = -\dfrac{\partial g}{\partial x}^T$

  local minima

- g has isolated local



- $\sigma(x) = g(x) - g(x^*)$ ← one of these local minimas

- $$\frac{\partial \sigma}{\partial x} f(x) = -\left|\frac{\partial g}{\partial x}\right|^2 \leq 0$$
$$\leq -c\,|x - x^*|^2$$

4. Input-output Analysis

- Motivating example:

we assume n_t is bounded

$$\alpha_{t+1} = \alpha_t - \tau \underset{Z \sim \mathcal{D}(\alpha_t)}{E}\left[\nabla_\alpha \ell(Z, \alpha_t)\right] + \tau n_t$$

to a nominal case : $\mathcal{D}(0)$

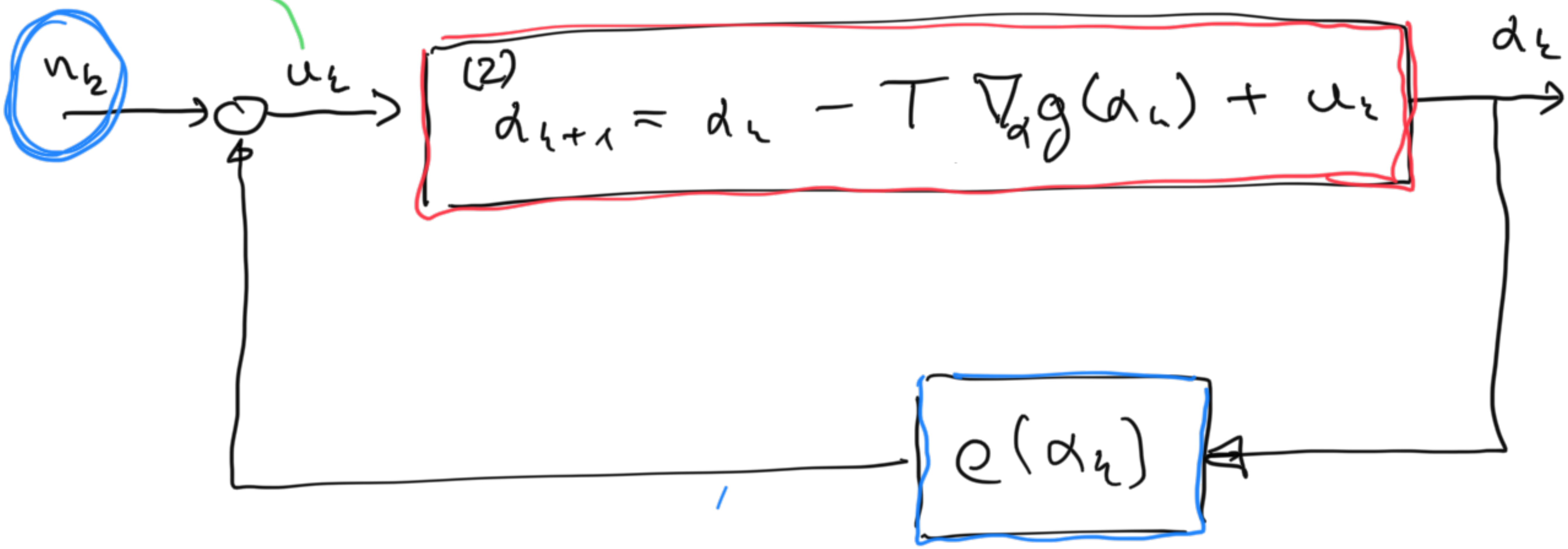
- We compare to

$$\alpha_{t+1} = \alpha_t - T \overbrace{\mathop{E}_{z\sim D(0)}\left[\nabla_\alpha l(z, \alpha_t) \right]}^{\nabla_\alpha g(\alpha_t)}$$

$$+ T \underbrace{\left(\mathop{E}_{z\sim D(0)}\left[\nabla_\alpha l(z, \alpha_t) \right] - \mathop{E}_{z\sim D(\alpha_t)}\left[\nabla_\alpha l(z, \alpha_t) \right] \right)}_{:= e(\alpha_t)}$$

$$+ T n_t$$

$u_t = n_t + e(\alpha_t)$



We will interpret (2) as a map from $u_t \to \alpha_t$

- We will interpret (2) as ...

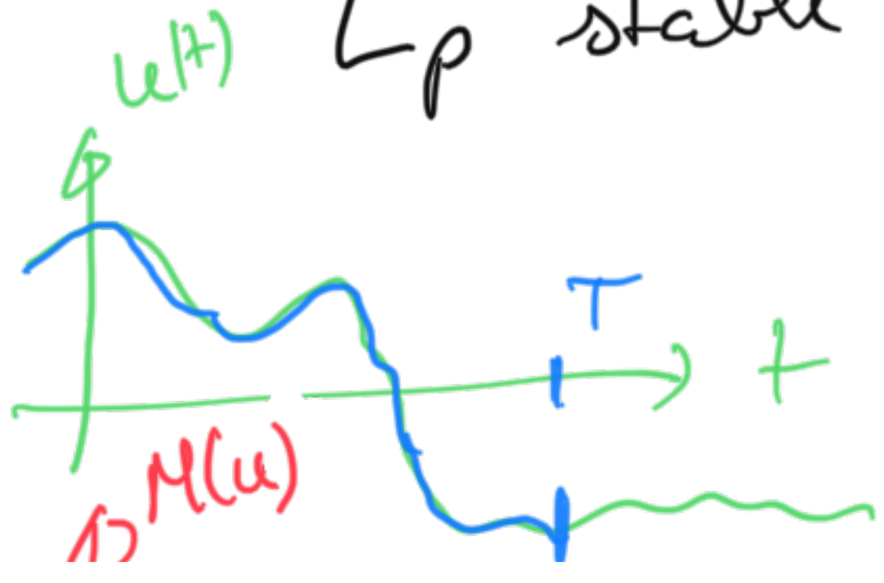
- $L_p := \{ g : [0,\infty) \to \mathbb{R}^n \mid \int_0^\infty |g(t)|^p \, dt < \infty \}$

- operator T : $$g_T(t) = \begin{cases} g(t) & t \le T \\ 0 & t > T \end{cases}$$

$$L_{pe} := \{ g : [0,\infty) \to \mathbb{R}^n \mid g_T \in L_p, \text{ for every } T \geq 0 \}.$$

$$g(t) = t^2, \; t \geq 0 \implies \begin{array}{l} g \in L_{2e} \\ g \notin L_2 \end{array}$$

- A mapping $H : L_{pe} \to L_{pe}$ is finite gain L_p stable if $\exists\, \gamma, \beta$:



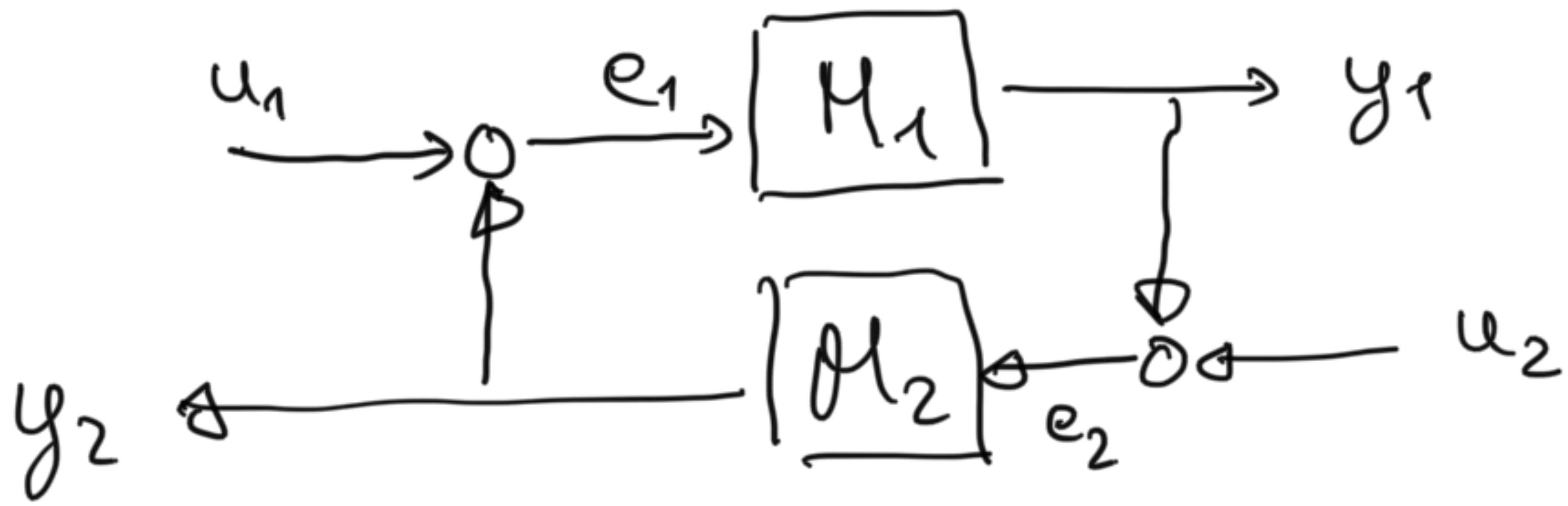
output

$$\|(H(u))_T\|_p \le \gamma \|u_T\|_p + \beta$$

contains the effect of the initial cond.

for all $u \in L_{pe}$ and $T \geq 0$.

- Thm

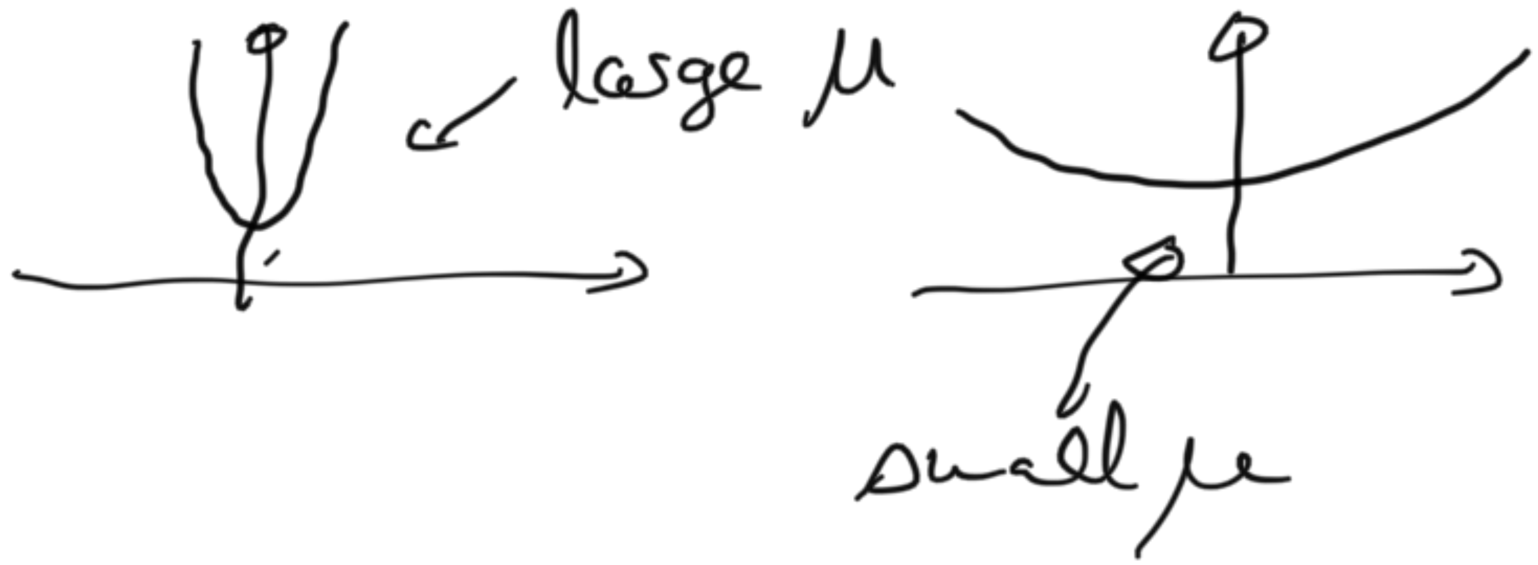


- Let the interconnection be well-defined (unique e_1, e_2, y_1, y_2). If H_1 and H_2 are finite gain L_p-stable with gain k_1, k_2 then the interconnection is finite gain L_p-stable provided that $k_1 k_2 < 1$.

- Example :

$\boxed{e(x_u)}$

GD is finite-gain L^∞-stable with gain $1/\mu$, μ strong convexity constant of g



if $e(x_u)$ is Lipschitz continuous with const. L_e

$\rightarrow$ interconnection is stable if $L_e/\mu < 1$.